JOURNAL OF COMPUTATIONAL PHYSICS **84**, 259–278 (1989)

# A Spectral-Difference Method for Two-Dimensional Viscous Flow

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Received March 20, 1987; revised October 19, 1988

We propose a spectral-difference method for the 2-dimensional vorticity equation with a periodic boundary condition in one direction. The solution satisfies a semidiscrete conservation law, and thus better numerical results are obtained. We also prove stability and convergence. © 1989 Academic Press, Inc.

## I. INTRODUCTION

Let $\xi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ be the vorticity and stream function, respectively. Let the coefficient of viscosity $\nu$ be positive. Let $\Omega = I \times \tilde{I}$, where

$$I = \{x_1 : 0 < x_1 < 1\}, \qquad \tilde{I} = \{x_2 : 0 < x_2 < 2\pi\},$$

and consider the problem

$$\begin{aligned}
&\frac{\partial \xi}{\partial t} + \frac{\partial \psi}{\partial x_2}\frac{\partial \xi}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial \xi}{\partial x_2} - \nu \nabla^2 \xi = f_1 \qquad \text{in} \quad \Omega \times (0, T],\\
&-\nabla^2 \psi = \xi + f_2 \qquad \text{in} \quad \Omega \times (0, T],\\
&\xi(x_1, x_2, t) = \xi(x_1, x_2 + 2\pi, t) \qquad \text{for} \quad t \geqslant 0, \qquad (1.1)\\
&\psi(x_1, x_2, t) = \psi(x_1, x_2 + 2\pi, t) \qquad \text{for} \quad t \geqslant 0,\\
&\xi(x_1, x_2, 0) = \xi_0(x_1, x_2) \qquad \text{in} \quad \bar{\Omega},
\end{aligned}$$

where

$$\begin{aligned}
&f_l(x_1, x_2 + 2\pi, t) = f_1(x_1, x_2, t) \qquad \text{for} \quad l = 1, 2,\\
&\xi_0(x_1, x_2 + 2\pi) = \xi_0(x_1, x_2).
\end{aligned}$$

For simplicity we assume that

$$\xi(0, x_2, t) = \xi(1, x_2, t) = \psi(0, x_2, t) = \psi(1, x_2, t) = 0.$$



0021-9991/89 $3.00

There is a lot of literature concerning finite element and finite difference methods to solve problem (1.1). But for any fixed scheme the accuracy of the solution is limited, even if the solution is infinitely smooth. In the past ten years the spectral method has been developed. See the references by Gottlieb and Orszag [1], Pasciak [2], Kreiss and Oliger [5], Ben-yu Guo [6], and He-ping Ma and Ben-yu Guo [7]. All of this work is for periodic problems, and thus it may not be applied to solve (1.1). On the other hand, Murdock [9, 10] and Vanel, Peyret, and Bontoux [11] used Chebyshev spectral methods to solve it. In this paper we follow the idea of [8] to construct a class of spectral-difference schemes for solving (1.1). The key point is the use of a skew symmetric decomposition of the nonlinear convection terms. If we choose the parameters suitably, then the numerical solution satisfies semidiscrete conservation laws. Generalized stability (see Ben-yu Guo [12] and Griffiths [13]) and convergence are proved. We find out that better error estimates are obtained by using the skew symmetric decomposition with suitable parameters.

## II. The Scheme and Conservation Laws

Let $h$ be the mesh spacing in the $x_1$-direction with $Mh=1$, and let

$$I_h=\{x_1=jh\colon 1\leqslant j\leqslant M-1\} \qquad \text{and} \qquad \Omega_h=I_h\times\tilde{I}.$$

Let $\tau$ be the mesh spacing in the $t$-direction, and let $S_\tau=\{t=k\tau\colon\ k=0,1,...\}$. Define

$$u_{x_1}(x_1,x_2,t)=\frac{1}{h}(u(x_1+h,x_2,t)-u(x_1,x_2,t)),$$

$$u_{\bar{x}_1}=u_{x_1}(x_1-h,x_2,t),$$

$$u_{\hat{x}_1}=\tfrac{1}{2}(u_{x_1}(x_1,x_2,t)+u_{\bar{x}_1}(x_1,x_2,t)),$$

$$\Delta u=u_{x_1\bar{x}_1}(x_1,x_2,t)+\frac{\partial^2 u}{\partial x_2^2}(x_1,x_2,t),$$

$$u_t=\frac{1}{\tau}(u(x_1,x_2,t+\tau)-u(x_1,x_2,t)).$$

The key problem in the construction of a reasonable scheme is to simulate as much as possible the properties of the solution of (1.1). Indeed, if $f_1=f_2=0$, then

$$\iint_\Omega \xi(x_1,x_2,t)\,dx_1\,dx_2-\nu\int_0^t\int_I\left(\frac{\partial\xi}{\partial x_1}(1,x_2,y)-\frac{\partial\xi}{\partial x_1}(0,x_2,y)\right)dx_2\,dy$$
$$=\iint_\Omega \xi_0(x_1,x_2)\,dx_1\,dx_2 \tag{2.1}$$

and

$$\iint_{\Omega} \xi^2(x_1, x_2, t)\, dx_1\, dx_2 + 2\nu \int_0^t \iint_{\Omega} \left[\left(\frac{\partial \xi}{\partial x_1}(x_1, x_2, y)\right)^2 + \left(\frac{\partial \xi}{\partial x_2}(x_1, x_2, y)\right)^2\right] dx_1\, dx_2\, dy = \iint_{\Omega} \xi_0^2(x_1, x_2)\, dx_1\, dx_2. \tag{2.2}$$

We shall construct a scheme, the solution of which satisfies semidiscrete conservation laws. Note that

$$\frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_1} - \frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_2} u\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial w}{\partial x_1} u\right) = \frac{\partial}{\partial x_2}\left(w \frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_1}\left(w \frac{\partial u}{\partial x_2}\right).$$

We therefore define

$$J_1(u, w) = \frac{\partial w}{\partial x_2} u_{\hat{x}_1} - w_{\hat{x}_1} \frac{\partial u}{\partial x_2},$$

$$J_2(u, w) = \left(\frac{\partial w}{\partial x_2} u\right)_{\hat{x}_1} - \frac{\partial}{\partial x_2}(w_{\hat{x}_1} u),$$

$$J_3(u, w) = \frac{\partial}{\partial x_2}(w u_{\hat{x}_1}) - \left(w \frac{\partial u}{\partial x_2}\right)_{\hat{x}_1},$$

and

$$J^{(\alpha)}(u, w) = \sum_{l=1}^{3} \alpha_l J_l(u, w),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, each $\alpha_l$ is positive, and $\sum \alpha_l = 1$.

Now let

$$V_N = \operatorname{span}\{\exp\{inx_2\} : |n| \leq N\},$$

and let $P_N$ be the orthogonal projection operation onto $V_N$, i.e.,

$$\int_I (P_N u - u)\bar{v}\, dx_2 = 0$$

for all $v \in V_N$. Let $\eta^{(N)}$ and $\phi^{(N)}$ be the approximations to $\xi$ and $\psi$, respectively, where

$$\eta^{(N)}(x_1, x_2, t) = \sum_{|n| \leq N} \eta_n^{(N)}(x_1, t) \exp\{inx_2\},$$

$$\phi^{(N)}(x_1, x_2, t) = \sum_{|n| \leq N} \phi_n^{(N)}(x_1, t) \exp\{inx_2\}.$$

The spectral-difference scheme for (1.1) is

$$\begin{aligned} \eta_t^{(N)} + P_N J^{(\alpha)}(\eta^{(N)} + \delta\tau\eta_t^{(N)}, \phi^{(N)}) - \nu\, \Delta(\eta^{(N)} + \sigma\tau\eta_t^{(N)}) &= P_N f_1 \quad \text{in } \Omega_h \times S_\tau, \\ -\Delta\phi^{(N)} &= \eta^{(N)} + P_N f_2 \quad \text{in } \Omega_h \times S_\tau, \\ \eta^{(N)}(0, x_2, t) = \eta^{(N)}(1, x_2, t) = \phi^{(N)}(0, x_2, t) &= \phi^{(N)}(1, x_2, t), \\ \eta^{(N)}(x_1, x_2, 0) = \eta_0^{(N)}(x_1, x_2) &= P_N \xi_0(x_1, x_2) \quad \text{in } \bar{\Omega}_h, \end{aligned} \tag{2.3}$$

where $\delta$ and $\sigma$ are parameters such that $0 \leq \delta,\ \sigma \leq 1$. If $\delta = \sigma = 0$, then (2.3) is an explicit scheme. Otherwise, we need iteration to get $\eta^{(N)}(x_1, x_2, t)$ for each $t \in S_\tau$.

Now we introduce some notations as follows:

$$(u(x_1), v(x_1))_I = \frac{1}{2\pi} \int_I u(x_1, x_2)\, \bar{v}(x_1, x_2)\, dx_2,$$

$$\|u(x_1)\|_I^2 = (u(x_1), u(x_1))_I,$$

$$(u, v) = h \sum_{x_1 \in I_h} (u(x_1), v(x_1))_I,$$

$$\|u\|^2 = (u, u),$$

$$|u|_1^2 = \frac{1}{2} \|u_{x_1}\|^2 + \frac{1}{2} \|u_{\bar{x}_1}\|^2 + \left\| \frac{\partial u}{\partial x_2} \right\|^2.$$

Assume $u = v = w = 0$ for $x_1 = 0$ or 1 and that $u$, $v$, and $w$ are periodic in $x_2$. From Abel's formula we obtain

$$(u_{\hat{x}_1}, v) + (v_{\hat{x}_1}, u) = 0, \tag{2.4}$$

$$\left(\frac{\partial u}{\partial x_2}, v\right) + \left(\frac{\partial v}{\partial x_2}, u\right) = 0, \tag{2.5}$$

which lead to

$$(J_1(u, w), 1) = \left(\frac{\partial w}{\partial x_2}, u_{\hat{x}_1}\right) + \left(\left(\frac{\partial w}{\partial x_2}\right)_{\hat{x}_1}, u\right) = 0, \tag{2.6}$$

$$(J_2(u, w), 1) = \left(\left(u \frac{\partial w}{\partial x_2}\right)_{\hat{x}_1}, 1\right) = A(u, w), \tag{2.7}$$

$$(J_3(u, w), 1) = -A(w, u) = A(u, w), \tag{2.8}$$

where

$$A(u, w)=\frac{1}{2}\left(u(1-h), \frac{\partial w}{\partial x_2}(1-h)\right)_{\bar{I}}-\frac{1}{2}\left(u(h), \frac{\partial w}{\partial x_2}(h)\right)_{\bar{I}}.$$

We have also from (2.4) and (2.5) that

$$\left(\frac{\partial w}{\partial x_2} u_{\hat{x}_1}, v\right)+\left(\left(\frac{\partial w}{\partial x_2} v\right)_{\hat{x}_1}, u\right)=0,$$

$$\left(w_{\hat{x}_1} \frac{\partial u}{\partial x_2}, v\right)+\left(\frac{\partial}{\partial x_2}(w_{\hat{x}_1} v), u\right)=0.$$

Thus, it follows that

$$(J_1(u, w), v)+(J_2(v, w), u)=0. \tag{2.9}$$

Similarly, we have

$$(J_3(u, w), v)+\left(\frac{\partial v}{\partial x_2}, w u_{\hat{x}_1}\right)-\left(\frac{\partial u}{\partial x_2}, w v_{\hat{x}_1}\right)=0. \tag{2.10}$$

From (2.6)–(2.10) we have

$$(J^{(\alpha)}(u, w), 1)=(\alpha_2+\alpha_3) A(u, w), \tag{2.11}$$

$$(J^{(\alpha)}(u, w), v)+(J^{(\alpha)}(v, w), u)=(\alpha_1-\alpha_2) \times[(J_2(v, w), u)+(J_2(u, w), v)]. \tag{2.12}$$

In particular, if $\alpha_1=\alpha_2$, then

$$(J^{(\alpha)}(u, w), u)=0.$$

It is easy to show that

$$(u, \Delta v)+\tfrac{1}{2}(u_{x_1}, v_{x_1})+\tfrac{1}{2}(u_{\bar{x}_1}, v_{\bar{x}_2}) +\left(\frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2}\right)+S(u, v)=0, \tag{2.13}$$

where

$$S(u, v)=\frac{1}{2h}(u(h), v(h))_{\bar{I}}+\frac{1}{2h}(u(1-h), v(1-h))_{\bar{I}}.$$

In particular, with the notation $S(u)=S(u, u)$ we have

$$(\Delta u, u)+|u|_1^2+S(u)=0. \tag{2.14}$$

We next check the conservation laws. Assume that $f_1 = f_2 = 0$. We first sum (2.3) over all $(x_1, x_2) \in \Omega_h$ to get from (2.11) and (2.13) that

$$(\eta^{(N)}(t), 1)_t + (\alpha_2 + \alpha_3)\, A(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \phi^{(N)}(t)) \\ - \nu S(\eta^{(N)}(t) + \sigma\tau\eta_t^{(N)}(t), 1) = 0,$$

and thus

$$(\eta^{(N)}(t), 1)) + \tau \sum_{\substack{y \in S_\tau \\ y \leqslant t-\tau}} [(\alpha_2 + \alpha_3)\, A\eta^{(N)}(y) + \delta\tau\eta_t^{(N)}(y), \phi^{(N)}(y)) \\ - \nu S(\eta^{(N)}(y) + \sigma\tau\eta_t^{(N)}(y), 1)] = (\eta^{(N)}(0), 1). \tag{2.15}$$

Second, we put $\alpha_1 = \alpha_2$ and $\delta = \sigma = \frac{1}{2}$ to get

$$\hat{\eta}^{(N)}(x_1, x_2, t) = \tfrac{1}{2}(\eta^{(N)}(x_1, x_2, t) + \eta^{(N)}(x_1, x_2, t+\tau)).$$

By taking the scalar product of the first formula of (2.3) with $2\hat{\eta}^{(N)}$, we have from (2.12), (2.13), and (2.14) that

$$\|\eta^{(N)}(t)\|_t^2 + 2\nu\, |\hat{\eta}^{(N)}(t)|_1^2 + 2\nu S(\hat{\eta}^{(N)}(t)) = 0;$$

thus

$$\|\eta^{(N)}(t)\|^2 + 2\nu\tau \sum_{\substack{y \in S_\tau \\ y \leqslant t-\tau}} [|\eta^{(N)}(t)|_1^2 + S(\hat{\eta}^{(N)}(t))] = \|\eta^{(N)}(0)\|^2. \tag{2.16}$$

Clearly, (2.15) and (2.16) are reasonable analogues of (2.1) and (2.2), respectively.

## III. Numerical Results

For convenience we take $\Omega = (0, 1) \times (0, 1)$ and $\delta = \sigma = 0$ in our computation. We deal with the problem with periodic boundary conditions in the $x_2$-direction and Dirichlet boundary conditions in the $x_1$-direction.

Let $\tilde{I}_h = \{x_2 : x_2 = jh,\ 0 \leqslant j \leqslant N-1\}$, and define

$$E_\infty(t) = \max_{(x_1, x_2) \in I_h \times \tilde{I}_h} |\xi(x_1, x_2, t) - \eta(x_1, x_2, t)|,$$

$$E_2(t) = \left(\frac{h}{N} \sum_{(x_1, x_2) \in I_h \times \tilde{I}_h} |\xi(x_1, x_2, t) - \eta(x_1, x_2, t)|^2\right)^{1/2}.$$

where $\eta(x_1, x_2, t)$ is the spectral-difference (or difference) approximation to $\xi(x_1, x_2, t)$.

In this section we list two tables for two kinds of flows. All of our experiments are for $\alpha_1 = \alpha_2$.

EXAMPLE 1. Let

$$\xi(x_1, x_2, t) = A \exp\{B \sin(2\pi x_2 + Cx_1) + \omega t\},$$
$$\psi(x_1, x_2, t) = A \exp\{\omega t\}(\sin 2\pi x_2 + Cx_1).$$

The numerical results are shown in Table I. These results are for scheme (2.3) at $t=1$ for $A=C=\omega=0.1$, $B=0.01$, and $\tau=\nu=0.001$. It is obvious that if we take $\alpha_1=\alpha_2$, then the solutions satisfy semidiscrete conservation laws, and better numerical results are obtained. Usually we take $\alpha_1=\alpha=\frac{1}{2}$ or $\alpha_1=\alpha_2=\alpha_3=\frac{1}{3}$ as in [3, 14]. Arakawa [14] also analyzed the advantages of such choices. Table I also shows that we get good results even for small $N$.

EXAMPLE 2. Let

$$\xi(x_1, x_2, t) = A \exp\{B \sin(2\pi x_2 + Cx_1) + \omega t\},$$
$$\psi(x_1, x_2, t) = A \exp\{\omega t\} \sin 2\pi x_2 \sin Cx_1.$$

We first use the spectral-difference scheme (2.3) to solve the 2-dimensional vorticity equation. For the sake of comparsion we use the difference scheme of [3] to solve the same problem. Let $\bar{h}=2\pi/\bar{M}$ and let $\Omega'_{\bar{h}}$ be the set of lattice points in $\Omega$. We define

$$u_{x_2}(x_1, x_2, t) = (1/\bar{h})(u(x_1, x_2+\bar{h}, t) - u(x_1, x_2, t)),$$
$$u_{\bar{x}_2}(x_1, x_2, t) = u_{x_2}(x_1, x_2-\bar{h}, t),$$
$$u_{\hat{x}_2}(x_1, x_2, t) = \tfrac{1}{2}(u_{x_2}(x_1, x_2, t) + u_{\bar{x}_2}(x_1, x_2, t)),$$
$$\Delta_{\bar{h}} u(x_1, x_2, t) = u_{x_1\bar{x}_1}(x_1, x_2, t) + u_{x_2\bar{x}_2}(x_1, x_2, t),$$

TABLE I

Errors for Scheme (2.3)

| | $M=10, N=4$ | | $M=10, N=8$ | |
|---|---|---|---|---|
| $(\alpha_1, \alpha_2, \alpha_2)$ | $E_2(t)\times 10^3$ | $E_\infty(t)\times 10^3$ | $E_2(t)\times 10^3$ | $E_\infty(t)\times 10^3$ |
| $(1, 0, 0)$ | 0.3460 | 0.8759 | 0.3484 | 0.8658 |
| $(\frac{1}{2}, \frac{1}{2}, 0)$ | 0.2217 | 0.6949 | 0.1906 | 0.5465 |
| $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | 0.2822 | 0.8696 | 0.1942 | 0.5118 |

TABLE II

Errors for Schemes (2.3) and (3.1)

| | Scheme (2.3) $M=10, N=4$ | Scheme (3.1) $M=10, \bar{M}=10$ |
|---|---|---|
| $(\alpha_1, \alpha_2, \alpha_2)$ | $E_2(t)\times 10^3$ | $E_2(t)\times 10^3$ |
| $(1, 0, 0)$ | 0.1753 | 0.2133 |
| $(\frac{1}{2}, \frac{1}{2}, 0)$ | 0.1621 | 0.2141 |
| $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | 0.1501 | 0.2138 |

and

$$\begin{aligned}
J_{1,\bar{h}}(u, w) &= w_{\hat{x}_2} u_{\hat{x}_1} - w_{\hat{x}_1} u_{\hat{x}_2}, \\
J_{2,\bar{h}}(u, w) &= (w_{\hat{x}_2} u)_{\hat{x}_1} - (w_{\hat{x}_1} u)_{\hat{x}_2}, \\
J_{3,\bar{h}}(u, w) &= (w u_{\hat{x}_1})_{\hat{x}_2} - (w u_{\hat{x}_2})_{\hat{x}_1}, \\
J_{\bar{h}}^{(\alpha)}(u, w) &= \sum_{l=1}^{3} \alpha_l J_{l,\bar{h}}(u, w),
\end{aligned}$$

where $\alpha_1+\alpha_2+\alpha_3=1$ and $\alpha_l \geqslant 0$ for $l=1, 2, 3$. Let $\eta^{\bar{h}}$ and $\phi^{\bar{h}}$ be the finite difference approximations to $\xi$ and $\psi$, respectively. The difference scheme is [3]

$$\begin{aligned}
\eta_t^{\bar{h}}(t) + J_{\bar{h}}^{(\alpha)}(\eta^{\bar{h}} + \delta\tau\eta_t^{\bar{h}}, \phi^{\bar{h}}) - \nu\, \Delta_{\bar{h}}(\eta^{\bar{h}} + \sigma\tau\eta_t^{\bar{h}}) &= f_1^{\bar{h}} \quad \text{in } \Omega'_{\bar{h}} \times S_\tau, \\
-\Delta_{\bar{h}}\phi^{\bar{h}} &= \eta^{\bar{h}} + f_2^{\bar{h}} \quad \text{in } \Omega'_{\bar{h}} \times S_\tau.
\end{aligned} \tag{3.1}$$

The numerical results by using schemes (2.3) and (3.1) with $\delta=\sigma=0$ are shown in Table II. These results are taken at $t=1$ for $A=B=C=\omega=0.1$ and $\tau=\nu=0.001$. It can be seen that the spectral-difference scheme (2.3) can give better results than the difference scheme (3.1).

## IV. Some Lemmas

In order to estimate the error, we need some lemmas.

Lemma 1. *For all $u(x_1, x_2, t)$ we have*

$$\begin{aligned}
2(u(t), u_t(t))_l &= (\|u(t)\|_l^2)_t - \tau\, \|u_t(t)\|_l^2, \\
2(u(t), u_t(t)) &= (\|u(t)\|^2)_t - \tau\, \|u_t(t)\|^2.
\end{aligned}$$

LEMMA 2. *If* $u(0, x_2, t) = u(1, x_2, t) = 0$ *and* $u(x_1, x_2, t) = u(x_1, x_2 + 2\pi, t)$, *then*

$$\begin{aligned} -2(u_t(t), \Delta u(t)) &= -2(u(t), \Delta u_t(t)) \\ &= [|u(t)|_1^2 + S(u(t))]_t - \tau\, |u_t(t)|_1^2 - \tau S(u_t). \end{aligned}$$

LEMMA 3. *If* $u \in V_N$ *for* $x_1 \in I_h$, *then*

$$\left\| \frac{\partial u}{\partial x_2} \right\|^2 \leqslant N^2 \, \|u\|^2.$$

*Proof.* Let

$$u(x_1, x_2) = \sum_{|n| \leqslant N} u_n(x_1) \exp\{inx_2\}.$$

Then

$$\frac{\partial u}{\partial x_2} = i \sum_{|n| \leqslant N} n u_n(x_1) \exp\{inx_2\},$$

and thus

$$\left\| \frac{\partial u(x_1)}{\partial x_2} \right\|_I^2 \leqslant N^2 \, \|u(x_1)\|_I^2, \qquad \left\| \frac{\partial u}{\partial x_2} \right\|^2 \leqslant N^2 \, \|u\|^2.$$

LEMMA 4. *If* $u(0, x_2, t) = u(1, x_2, t) = 0$, *then*

$$\|u_{x_1}\|^2 \leqslant \frac{4}{h^2} \, \|u\|^2, \qquad \|u_{\bar{x}_1}\|^2 \leqslant \frac{4}{h^2} \, \|u\|^2.$$

*Proof.* We prove the first conclusion. Because

$$\begin{aligned} |u_{x_1}(x_1, x_2)|^2 &= \frac{1}{h^2} \, |u(x_1 + h, x_2) - u(x_1, x_2)|^2 \\ &\leqslant \frac{2}{h^2} \, (|u(x_1 + h, x_2)|^2 + |u(x_1, x_2)|^2), \end{aligned}$$

it follows that

$$\|u_{x_1}(x_1)\|_I^2 \leqslant \frac{2}{h^2} \, (\|u(x_1 + h)\|_I^2 + \|u(x_1)\|_I^2), \quad \|u_{x_1}\|^2 \leqslant \frac{4}{h^2} \, \|u\|^2.$$

LEMMA 5. *If for all* $x_1 \in I_h$ *we have* $u(x_1, x_2)$ *and* $v(x_1, x_2) \in V_N$, *then*

$$\|uv\|^2 \leqslant \frac{2N+1}{h} \, \|u\|^2 \, \|v\|^2.$$

*Proof.* Let

$$u(x_1, x_2) = \sum_{|n| \leqslant N} u_n(x_1) \exp\{inx_2\},$$

$$v(x_1, x_2) = \sum_{|n| \leqslant N} v_n(x_1) \exp\{inx_2\}.$$

Then we have from [7] that

$$\|u(x_1)\, v(x_1)\|_{\tilde{I}}^2 \leqslant (2N+1)\, \|u(x_1)\|_{\tilde{I}}^2\, \|v(x_1)\|_{\tilde{I}}^2.$$

From Jensen's inequality we obtain

$$\begin{aligned}
\|uv\|^2 &= h \sum_{x_1 \in I_h} \|u(x_1)\, v(x_1)\|_{\tilde{I}}^2 \\
&\leqslant h(2N+1) \sum_{x_1 \in I_h} \|u(x_1)\|_{\tilde{I}}^2\, \|v(x_1)\|_{\tilde{I}}^2 \\
&\leqslant h(2N+1) \sum_{x_1 \in I_h} \|u(x_1)\|_{\tilde{I}}^2 \sum_{x_1 \in I_h} \|v(x_1)\|_{\tilde{I}}^2 \\
&= \frac{2N+1}{h}\, \|u\|^2\, \|v\|^2.
\end{aligned}$$

LEMMA 6. *If $u(x_1, x_2) \in V_N$ for $x_1 \in I_h$ and $u(0, x_2) = u(1, x_2) = 0$, then $\|u\|^2 \leqslant C_1(|u|_1^2 + S(u))$, where $C_1$ is a positive constant depending on $\Omega_h$.*

*Proof.* We consider the eigenvalue problem

$$\begin{gathered}
-\Delta u = \lambda u \qquad \text{in} \quad \Omega_h, \\
u(x_1, x_2) = u(x_1, x_2 + 2\pi) \qquad \text{in} \quad \bar{\Omega}_h, \\
u(0, x_2) = u(1, x_2) = 0 \qquad \text{for} \quad x_2 \in \tilde{I}.
\end{gathered}$$

By taking the scalar product of the above equation with $u$, we have from (2.14) that

$$|u|_1^2 + S(u) = \lambda\, \|u\|^2,$$

and thus

$$\|u\|^2 \leqslant \frac{1}{\min \lambda} (|u|_1^2 + S(u)).$$

LEMMA 7 [15]. *If $0 \leqslant \mu \leqslant \beta$ and $u \in H^{\beta}(\tilde{I})$, then*

$$\|P_N u - u\|_{H^{\mu}(\tilde{I})} \leqslant CN^{\mu-\beta} \|u\|_{H^{\beta}(\tilde{I})}, \quad \|P_N u\|_{H^{\mu}(\tilde{I})} \leqslant C \|u\|_{H^{\mu}(\tilde{I})}.$$

LEMMA 8 [3]. *Assume that the following conditions are satisfied:*

(1) *$Z(t)$ is a nonnegative function defined on $S_\tau$.*

(2) *$\rho$, $a$, $b$, $M_1$, $M_2$, and $M_3$ are nonnegative constants.*

(3) *$H(Z)$ is a function such that if $Z \leqslant M_3$, then $H(Z) \leqslant 0$.*

(4) *For all $t \in S_\tau$,*

$$Z(t) \leqslant \rho + \tau \sum_{\substack{y \in S_\tau \\ y \leqslant t-\tau}} [M_1 Z(y) + M_2 N^a h^{-b} Z^2(y) + H(Z(y))].$$

(5) *$Z(0) \leqslant \rho$ and*

$$\rho \exp\{(M_1 + M_2)t\} \leqslant \min(M_3, h^b/N^a).$$

*Then for all $t \leqslant T$ we have*

$$Z(t) \leqslant \rho \exp\{(M_1 + M_2)t\}.$$

*In particular, if $M_2 = 0$ and $H(Z) \leqslant 0$, then for all $\rho$ and $t$, $Z(t) \leqslant \rho \exp\{M_1 t\}$.*

## V. ERROR ESTIMATION

Let $\mathscr{B}$ be a Banach space and let

$$|||u|||_{\mathscr{B}} = \max_{0 \leqslant t \leqslant T} \|u(t)\|_{\mathscr{B}}.$$

Define

$$\|u(t)\|_\infty = \max_{x_1 \in I_h, x_2 \in \tilde{I}_h} |u(x_1, x_2, t)|,$$

$$|u(t)|_{1,\infty} = \max_{x_1 \in I_h, x_2 \in \tilde{I}_h} \left( |u_{x_1}(x_1, x_2, t)|, |u_{\bar{x}_1}(x_1, x_2, t)|, \left|\frac{\partial u}{\partial x_2}(x_1, x_2, t)\right| \right)$$

$$\|u(t)\|_{1,\infty} = \|u(t)\|_\infty + |u(t)|_{1,\infty},$$

$$|||u|||_{1,\infty} = \max_{t \leqslant T} \|u(t)\|_{1,\infty}.$$

Assume that $\alpha_1 = \alpha_2$, that $\tau = O(h^2)$, that $\tau = O(1/N^2)$ and that

$$\tilde{\eta}^{(N)}(0, x_2, t) = \tilde{\eta}^{(N)}(1, x_2, t) = \tilde{\phi}^{(N)}(0, x_2, t) = \tilde{\phi}^{(N)}(1, x_2, t) = 0.$$

If $f_1, f_2$, and $\xi_0$ have the errors $\tilde{f}_1, \tilde{f}_2$, and $\tilde{\xi}_0$, respectively, then we get the solution $\bar{\eta}^{(N)}$ and $\bar{\phi}^{(N)}$ satisfying

$$\begin{aligned}
&\bar{\eta}_t^{(N)} + P_N J^{(\alpha)}(\bar{\eta}^{(N)} + \delta\tau\bar{\eta}_t^{(N)}, \bar{\phi}^{(N)}) \\
&\qquad - \nu\,\Delta(\bar{\eta}^{(N)} + \sigma\tau\bar{\eta}_t^{(N)}) = P_N(f_1 + \tilde{f}_1) \quad \text{in} \quad \Omega_h \times S_\tau, \\
&\qquad\qquad -\Delta\bar{\phi}^{(N)} = \bar{\eta}^{(N)} + P_N(f_2 + \tilde{f}_2) \quad \text{in} \quad \Omega_h \times S_\tau, \\
&\qquad\qquad \bar{\eta}^{(N)}(x_1, x_2, 0) = P_N(\xi_0(x_1, x_2) + \tilde{\xi}_0(x_1, x_2)) \quad \text{in} \quad \bar{\Omega}_h.
\end{aligned}$$

Let $\tilde{\eta}^{(N)} = \bar{\eta}^{(N)} - \eta^{(N)}$ and $\tilde{\phi}^{(N)} = \bar{\phi}^{(N)} - \phi^{(N)}$. Then it follows that

$$\begin{aligned}
&\tilde{\eta}_t^{(N)} + P_N J^{(\alpha)}(\tilde{\eta}^{(N)} + \delta\tau\tilde{\eta}_t^{(N)}, \phi^{(N)} + \tilde{\phi}^{(N)}) \\
&+ P_N J^{(\alpha)}(\eta^{(N)} + \sigma\tau\eta_t^{(N)}, \tilde{\phi}^{(N)}) - \nu\,\Delta(\tilde{\eta}^{(N)} + \sigma\tau\tilde{\eta}_t^{(N)}) = P_N\tilde{f}_1 \quad \text{in} \quad \Omega_h \times S_\tau, \\
&\qquad -\Delta\tilde{\phi}^{(N)} = \tilde{\eta}^{(N)} + P_N\tilde{f}_2 \quad \text{in} \quad \Omega_h \times S_\tau, \\
&\qquad \tilde{\eta}(N)(x_1, x_2, 0) = P_N\tilde{\xi}_0(x_1, x_2) \quad \text{in} \quad \bar{\Omega}_h. \qquad (5.1)
\end{aligned}$$

By taking the scalar product of the first equation of (5.1) with $2\tilde{\eta}^{(N)}$, we get from Lemmas 1 and 2 and from Eqs. (2.12) and (2.13) that

$$\begin{aligned}
&\|\tilde{\eta}^{(N)}(t)\|_t^2 - \tau\,\|\tilde{\eta}_t^{(N)}(t)\|^2 - 2\delta\tau(\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\tilde{\eta}^{(N)}(t), \tilde{\phi}^{(N)}(t))) \\
&\quad + 2(\tilde{\eta}(t), J^{(\alpha)}(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \tilde{\phi}^{(N)}(t))) \\
&\quad - 2\delta\tau(\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\tilde{\eta}^{(N)}(t), \phi^{(N)}(t))) \\
&\quad + 2\nu(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))) \\
&\quad + \nu\sigma\tau(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))_t) \\
&\quad - \nu\sigma\tau^2(|\tilde{\eta}_t^{(N)}(t)|_1^2 + S(\tilde{\eta}_t^{(N)}(t))) \\
&= 2(\tilde{\eta}^{(N)}(t), \tilde{f}_1(t)). \qquad (5.2)
\end{aligned}$$

Let $m$ be a positive constant which will be determined later. By taking the scalar product of the first equation of (5.1) with $m\tau\tilde{\eta}_t^{(N)}(t)$, we obtain from Lemmas 1 and 2 and from (2.12) that

$$\begin{aligned}
&m\tau\,\|\tilde{\eta}_t^{(N)}(t)\|^2 + m\tau(\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\tilde{\eta}^{(N)}(t), \tilde{\phi}^{(N)}(t))) \\
&\quad + m\tau(\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\tilde{\eta}^{(N)}(t), \phi^{(N)}(t))) \\
&\quad + m\tau(\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\eta^{(N)}(t), \tilde{\phi}^{(N)}(t))) \\
&\quad + \frac{m\nu\tau}{2}(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t)))_t \\
&\quad + m\nu\tau^2(\sigma - \tfrac{1}{2})(|\tilde{\eta}_t^{(N)}(t)|_1^2 + S(\tilde{\eta}_t^{(N)}(t))) \\
&= m\tau(\tilde{\eta}_t^{(N)}(t), \tilde{f}_1(t)). \qquad (5.3)
\end{aligned}$$

Let $\varepsilon > 0$ and let $C$ be a positive constant which may be different in different formulas. Putting (5.2) and (5.3) together, we get

$$\begin{aligned}
&\|\tilde{\eta}^{(N)}(t)\|^2 + \tau(m-1-\varepsilon)\,\|\tilde{\eta}_t^{(N)}(t)\|^2 + 2\nu(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t)))\\
&\quad + \nu\tau\left(\sigma + \frac{m}{2}\right)(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t)))_t\\
&\quad + \nu\tau^2\left(m\sigma - \sigma - \frac{m}{2}\right)(|\tilde{\eta}_t^{(N)}(t)|_1^2 + S(\tilde{\eta}_t^{(N)}(t)))\\
&\quad + \sum_{l=1}^{3} G_l(t) \leqslant \|\tilde{\eta}(t)\|^2 + \left(1 + \frac{m^2\tau}{4\varepsilon}\right)\|\tilde{f}_1(t)\|^2,
\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
G_1(t) &= (2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}^{(N)}(t), J^{(\alpha)}(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \tilde{\phi}^{(N)}(t))),\\
G_2(t) &= \tau(m-2\delta)(\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\tilde{\eta}^{(N)}(t), \phi^{(N)}(t))),\\
G_3(t) &= \tau(m-2\delta)(\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\tilde{\eta}^{(N)}(t), \tilde{\phi}^{(N)}(t))).
\end{aligned}$$

By taking the scalar product of the second formula of (5.1) with $\tilde{\phi}^{(N)}(t)$, we have from (2.14) that

$$\begin{aligned}
&|\tilde{\phi}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))\\
&\quad \leqslant \frac{1}{2C_1}\|\tilde{\phi}^{(N)}(t)\|^2 + \frac{C_1}{2}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2).
\end{aligned}$$

From this inequality and from Lemma 6 we conclude that

$$|\tilde{\phi}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t)) \leqslant C_1(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{5.5}$$

We are now going to estimate the terms $|G_l(t)|$. It is easy to verify that

$$\begin{aligned}
&|(\tilde{\eta}^{(N)}(t), J^{(\alpha)}(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \tilde{\phi}^{(N)}(t)))|\\
&\quad \leqslant C\,|||\eta^{(N)}|||_{1,\infty}^2\,|\tilde{\phi}^{(N)}(t)|_1^2,\\
&|m\tau(\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t), \tilde{\phi}^{(N)}(t)))|\\
&\quad \leqslant \varepsilon\tau\,\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C\tau m^2}{4\varepsilon}\,|||\eta^{(N)}|||_{1,\infty}^2\,|\tilde{\phi}^{(N)}(t)|_1^2.
\end{aligned}$$

Hence (5.5) leads to

$$\begin{aligned}
|G_1(t)| &\leqslant \varepsilon\tau\,\|\tilde{\eta}_t^{(N)}(t)\|^2 + C\left(1 + \frac{m^2\tau}{4\varepsilon}\right)\\
&\quad \times |||\eta^{(N)}|||_{1,\infty}^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2).
\end{aligned} \tag{5.6}$$

By the $\varepsilon$-inequality we have

$$|G_2(t)| \leqslant \varepsilon\tau \, \|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C\tau(m-2\delta)^2}{4\varepsilon} \, |||\phi|||_{1,\infty}^2 \, |\tilde{\eta}^{(N)}(t)|_1^2. \tag{5.7}$$

By Lemma 5 and (5.5) we obtain

$$\begin{aligned} |G_3(t)| &\leqslant \varepsilon\tau \, \|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{CN\tau(m-2\delta)^2}{4\varepsilon h} \, |\tilde{\eta}^{(N)}(t)|_1^2 \, |\tilde{\phi}^{(N)}(t)|_1^2 \\ &\leqslant \varepsilon\tau \, \|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{CN\tau(m-2\delta)^2}{4\varepsilon h} \\ &\quad \times |\tilde{\eta}^{(N)}(t)|_1^2 \, (\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \end{aligned} \tag{5.8}$$

By substituting (5.6)–(5.8) into (5.4), we obtain

$$\begin{aligned} &\|\tilde{\eta}^{(N)}(t)\|_t^2 + \tau(m-1-4\varepsilon) \, \|\tilde{\eta}_t^{(N)}(t)\|^2 + v(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))) \\ &\quad + v\tau\left(\sigma + \frac{m}{2}\right)(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t)))_t \\ &\quad + v\tau^2\left(m\sigma - \sigma - \frac{m}{2}\right)(|\tilde{\eta}_t^{(N)}(t)|_1^2 + S(\tilde{\eta}_t^{(N)}(t))) \\ &\leqslant H_0 \, \|\tilde{\eta}^{(N)}(t)\|^2 + H_1(t) \, |\tilde{\eta}^{(N)}(t)|_1^2 + R^{(N)}(t), \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} H_0 &= 1 + C\left(1 + \frac{m^2\tau}{4\varepsilon}\right) |||\eta|||_{1,\infty}^2, \\ H_1(t) &= -v + \frac{C\tau(m-2\delta)^2}{4\varepsilon} \\ &\quad \times \left(\frac{N}{h}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2) + |||\phi|||_{1,\infty}^2\right), \\ R^{(N)}(t) &= C\left(1 + \frac{m^2\tau}{4\varepsilon}\right)(\|\tilde{f}_1(t)\|^2 + |||\eta^{(N)}|||_{1,\infty}^2 \, \|\tilde{f}_2(t)\|^2). \end{aligned}$$

Let $\varepsilon$ be suitably small, and choose the value of $m$ as follows.

*Case* 1. $\sigma > \frac{1}{2}$. In this case we take

$$m > m_1 = \max\left(\frac{2\sigma}{2\sigma - 1}, 1 + p_0 + 4\varepsilon\right),$$

where $p_0 \geqslant 0$. Then (5.9) leads to

$$\|\tilde{\eta}^{(N)}(t)\|_t^2 + p_0\tau\, \|\tilde{\eta}_t^{(N)}(t)\|^2 + v\, |\tilde{\eta}^{(N)}(t)|_1^2 + v\tau\left(\sigma + \frac{m}{2}\right)[|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))]_t \leqslant H_0\, \|\tilde{\eta}^{(N)}(t)\|^2 + H_1(t)\, |\tilde{\eta}^{(N)}(t)|_1^2 + R^{(N)}(t). \tag{5.10}$$

*Case* 2. $\sigma = \frac{1}{2}$. In this case we take

$$m > m_2 = 1 + p_0 + \frac{1}{2} v\tau N^2 + \frac{9v\tau}{4h^2} + 4\varepsilon.$$

From Lemmas 3 and 4 and from the fact that

$$S(\tilde{\eta}_t^{(N)}(t)) \leqslant \frac{1}{h^2}\, \|\tilde{\eta}_t^{(N)}\|^2,$$

we have

$$\tau(m-1-4\varepsilon)\, \|\tilde{\eta}_t^{(N)}(t)\|^2 + v\tau^2\left(m\sigma - \sigma - \frac{m}{2}\right)[|\tilde{\eta}_t^{(N)}(t)|_1^2 + S(\tilde{\eta}_t^{(N)}(t))] \geqslant p_0\tau\, \|\tilde{\eta}_t^{(N)}(t)\|^2. \tag{5.11}$$

Thus, (5.10) also holds in this case.

*Case* 3. $\sigma < \frac{1}{2}$. In this case we also impose the condition that

$$\tau < \frac{4h^2}{v(1-2\sigma)(9+2N^2h^2)}.$$

Then if we take

$$m > m_3 = \left(1 + p_0 + \frac{9v\sigma\tau}{2h^2} + v\sigma\tau N^2 + 4\varepsilon\right) \times \left(1 - \frac{v\tau(9+2N^2h^2)(1-2\sigma)}{4h^2}\right)^{-1},$$

we get (5.11) and consequently (5.10).

Now let

$$\tilde{E}^{(N)}(t) = \|\tilde{\eta}^{(N)}(t)\|^2 + \tau \sum_{\substack{y \in S_\tau \\ y \leqslant t-\tau}} (p_0\tau\, \|\tilde{\eta}_t^{(N)}(y)\|^2 + v\, |\tilde{\eta}^{(N)}(y)|_1^2),$$

$$\rho^{(N)}(t) = \|\tilde{\eta}^{(N)}(0)\|^2 + \tau \sum_{\substack{y \in S_\tau \\ y \leqslant t-\tau}} \|R^{(N)}(y)\|^2.$$

By summing (5.10) we get

$$\tilde{E}^{(N)}(t) \leqslant \rho^{(N)}(t) + \tau \sum_{\substack{y \in S_\tau \\ y \leqslant t-\tau}} [H_0 \tilde{E}^{(N)}(y) + H_1(y) \, |\tilde{\eta}^{(N)}(y)|_1^2].$$

In particular, if

$$2\delta \geqslant \begin{cases} m_1 & \text{for } \sigma > \frac{1}{2}, \\ m_2 & \text{for } \sigma = \frac{1}{2}, \\ m_3 & \text{for } \sigma < \frac{1}{2}, \end{cases} \tag{5.12}$$

then we may take $m = 2\delta$, and so $H_1(t) = -\nu < 0$. Finally, we apply Lemma 8 to get the following result.

THEOREM 1. *Suppose that the following conditions are fulfilled:*

(1) $\alpha_1 = \alpha_2$, $\tau = O(h^2)$, *and* $\tau = O(1/N^2)$,

(2) $\sigma \geqslant \frac{1}{2}$ *or*

$$\tau < \frac{4h^2}{\nu(1-2\sigma)(9+2N^2h^2)},$$

(3) *for all* $t \leqslant T$ *we have*

$$C\tau(m-2\delta)^2 \, (Nh^{-1} \, \|\tilde{f}_2(t)\|^2 + |||\phi|||_{1,\infty}^2) < \nu\varepsilon,$$

(4) *for all* $t \leqslant T$ *we have*

$$\rho^{(N)}(t) e^{H_0 t} \leqslant \frac{2\varepsilon\nu}{C(m-2\delta)^2}.$$

*Then for* $t \leqslant T$ *we have*

$$\tilde{E}^{(N)}(t) \leqslant \rho^{(N)}(t) e^{H_0 t}.$$

*In particular, if* (5.12) *holds, then the above estimate holds for all* $\rho^{(N)}(t)$ *and* $t$.

*Remark.* Since we have taken $\alpha_1 = \alpha_2$, the main nonlinear error terms disappear, i.e.,

$$(\tilde{\eta}^{(N)}(t), J^{(\alpha)}(\tilde{\eta}^{(N)}(t), \phi^{(N)}(t))) = (\tilde{\eta}_t^{(N)}(t), J^{(\alpha)}(\tilde{\eta}_t^{(N)}(t), \tilde{\phi}^{(N)}(t))) = 0.$$

If this were not the case, we would need to replace condition (4) by

$$\rho^{(N)}(t) e^{H_0 t} \leqslant \frac{2\varepsilon\nu h}{C(m-2\delta)^2}.$$

## VI. CONVERGENCE

We now consider the question of convergence. Let $\xi^{(N)}=P_N\xi$, $\psi^{(N)}=P_N\psi$, $\tilde{\xi}^{(N)}=\eta^{(N)}-\xi^{(N)}$, and $\tilde{\psi}^{(N)}=\phi^{(N)}-\psi^{(N)}$. Then we have

$$\xi_t^{(N)}+P_N J^{(\alpha)}(\xi^{(N)}+\delta\tau\xi_t^{(N)},\psi^{(N)})$$
$$-\nu\,\Delta(\xi^{(N)}+\sigma\tau\xi_t^{(N)})=P_N f_1+\sum_{l=1}^{5}M_l^{(N)}\qquad\text{in}\quad \Omega_h\times S_\tau,$$
$$-\Delta\psi^{(N)}=\xi^{(N)}+P_N f_2+M_6^{(N)}\qquad\text{in}\quad \Omega_h\times S_\tau,$$
$$\xi^{(N)}(0,x_2,t)=\xi^{(N)}(1,x_2,t)=\psi^{(N)}(0,x_2,t)=\psi^{(N)}(1,x_2,t)=0,$$
$$\xi^{(N)}(x_1,x_2,0)=P_N\xi_0(x_1,x_2)\qquad\text{in}\quad \bar{\Omega}_h,$$

where

$$M_1^{(N)}=\xi_t^{(N)}-\frac{\partial\xi^{(N)}}{\partial t},$$
$$M_2^{(N)}=P_N J^{(\alpha)}(\xi^{(N)},\psi^{(N)})-P_N\left(\frac{\partial\psi}{\partial x_2}\frac{\partial\xi}{\partial x_1}-\frac{\partial\psi}{\partial x_1}\frac{\partial\xi}{\partial x_2}\right).$$
$$M_3^{(N)}=\delta\tau P_N J^{(\alpha)}(\xi_t^{(N)},\psi^{(N)}),$$
$$M_4^{(N)}=\nu\frac{\partial^2\xi^{(N)}}{\partial x_1^2}-\nu\xi_{x_1\bar{x}_1}^{(N)},$$
$$M_5^{(N)}=\nu\sigma\tau\,\Delta\xi_t^{(N)},$$
$$M_6^{(N)}=\frac{\partial^2\psi^{(N)}}{\partial x_1^2}-\psi_{x_1\bar{x}_1}^{(N)}.$$

Furthermore, we have

$$\tilde{\xi}_t^{(N)}+P_N J^{(\alpha)}(\tilde{\xi}^{(N)}+\sigma\tau\tilde{\xi}_t^{(N)},\psi^{(N)}+\tilde{\psi}^{(N)})+P_N J^{(\alpha)}(\xi^{(N)}+\delta\tau\xi_t^{(N)},\tilde{\psi}^{(N)})$$
$$-\nu\Delta(\tilde{\xi}^{(N)}+\sigma\tau\tilde{\xi}_t^{(N)})=-\sum_{l=1}^{5}M_l^{(N)},\qquad\text{in}\quad \Omega_h\times S_\tau,$$
$$-\Delta\tilde{\psi}^{(N)}=\tilde{\xi}^{(N)}-M_6^{(N)},\qquad\text{in}\quad \Omega_h\times S_\tau,$$
$$\tilde{\xi}^{(N)}(0,x_2,t)=\tilde{\xi}^{(N)}(1,x_2,t)=\tilde{\psi}^{(N)}(0,x_2,t)=\tilde{\psi}^{(N)}(1,x_2,t)=0,$$
$$\tilde{\xi}^{(N)}(x_1,x_2,0)=0\qquad\text{in}\quad \bar{\Omega}_h.$$

We now estimate $|M_l^{(N)}(t)|$. First, we have from Lemma 1 that

$$\|M_1(t)\| = \left\|P_N\left(\frac{\partial \xi}{\partial t}(t) - \xi_t(t)\right)\right\| \leqslant C\left\|\frac{\partial \xi}{\partial t}(t) - \xi_t(t)\right\| \leqslant C\tau \left\|\left\|\frac{\partial^2 \xi}{\partial t^2}\right\|\right\|_{C(I,L^2(\bar{I}))}.$$

We have

$$M_2(t) = \sum_{l=1}^{3} M_2^{(l)}(t),$$

where

$$M_2^{(1)} = P_N\left(J^{(\alpha)}(\xi^{(N)}, \psi^{(N)}) - \frac{\partial \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1} + \frac{\partial \psi^{(N)}}{\partial x_1}\frac{\partial \xi^{(N)}}{\partial x_1}\right),$$

$$M_2^{(2)} = P_N\left(\frac{\partial \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1} - \frac{\partial \psi^{(N)}}{\partial x_1}\frac{\partial \xi^{(N)}}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1} + \frac{\partial \psi}{\partial x_1}\frac{\partial \xi^{(N)}}{\partial x_2}\right),$$

$$M_2^{(3)} = P_N\left(\frac{\partial \psi}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial \xi^{(N)}}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial \xi}{\partial x_1} + \frac{\partial \psi}{\partial x_1}\frac{\partial \xi}{\partial x_2}\right).$$

Let $\mu > 0$ and $r > 0$. By Lemma 1 and embedding theory we get

$$\begin{aligned}
\|M_2^{(1)}(t)\| &\leqslant Ch^2 \|\psi^{(N)}(t)\|_{C^1(\Omega)} \|\xi^{(N)}(t)\|_{C^3(I,L^2(\bar{I}))} \\
&\leqslant Ch^2 \|\psi^{(N)}(t)\|_{H^{2+r}(\Omega)} \|\xi^{(N)}(t)\|_{H^{7/2+r}(I,L^2(\bar{I}))} \\
&\leqslant Ch^2 \|\psi(t)\|_{H^{2+r}(\Omega)} \|\xi(t)\|_{H^{7/2+r}(I,L^2(\bar{I}))} \\
&\leqslant Ch^2 \|\|\psi\|\|_{H^{2+r}(\Omega)} \|\|\xi\|\|_{H^{7/2+r}(I,L^2(\bar{I}))},
\end{aligned}$$

$$\begin{aligned}
\|M_2^{(2)}\| &\leqslant C\left(\left\|\left(\frac{\partial \psi^{(N)}}{\partial x_2}(t) - \frac{\partial \psi}{\partial x_2}(t)\right)\frac{\partial \xi^{(N)}}{\partial x_1}(t)\right\| + \left\|\left(\frac{\partial \psi^{(N)}}{\partial x_1}(t) - \frac{\partial \psi}{\partial x_1}(t)\right)\frac{\partial \xi^{(N)}}{\partial x_2}(t)\right\|\right) \\
&\leqslant CN^{-\mu}(\|\psi(t)\|_{C^1(I,H^\mu(\bar{I}))} + \|\psi(t)\|_{C^1(I,H^{\mu+1}(\bar{I}))}) \|\xi^{(N)}(t)\|_{C^1(\Omega)} \\
&\leqslant CN^{-\mu}(\|\|\psi\|\|_{H^{3/2+r}(I,H^\mu(\bar{I}))} + \|\|\psi\|\|_{H^{1/2+r}(I,H^{\mu+1}(\bar{I}))}) \|\|\xi\|\|_{H^{2+r}(\Omega)}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\|M_2^{(3)}(t)\| &\leqslant CN^{-\mu} \|\psi^{(N)}(t)\|_{C^1(\Omega)} \\
&\quad \times (\|\xi^{(N)}(t)\|_{C^1(I,H^\mu(\bar{I}))} + \|\xi(t)\|_{C(I,H^{\mu+1}(\bar{I}))}) \\
&\leqslant CN^{-\mu} \|\|\psi\|\|_{H^{2+r}(\Omega)} \\
&\quad \times (\|\|\xi\|\|_{H^{3/2+r}(I,H^\mu(\bar{I}))} + \|\|\xi\|\|_{H^{1/2+r}(I,H^{\mu+1}(\bar{I}))}).
\end{aligned}$$

By the embedding theorem we also have

$$\begin{aligned}\|M_3(t)\| &\leqslant C\tau \, \|J^{(\alpha)}(\xi_t^{(N)}(t), \psi_t^{(N)}(t))\| \\ &\leqslant C\tau \, \|\psi^{(N)}(t)\|_{C^1(\Omega)} \\ &\quad \times (\|\xi_t^{(N)}(t)\|_{C^1(I, L^2(\tilde{I}))} + \|\xi_t^{(N)}(t)\|_{C(I, H^1(\tilde{I}))}) \\ &\leqslant C\tau \, |||\psi|||_{H^{2+r}(\Omega)} \\ &\quad \times \left( \left|\left|\left| \frac{\partial \xi}{\partial t} \right|\right|\right|_{H^{3/2+r}(I, L^2(\tilde{I}))} + \left|\left|\left| \frac{\partial \xi}{\partial t} \right|\right|\right|_{H^{1/2+r}(I, H^1(\tilde{I}))} \right).\end{aligned}$$

Clearly, we also have

$$\begin{aligned}\|M_4(t)\| &\leqslant C \left\| \frac{\partial^2 \xi}{\partial x_1^2}(t) - \xi_{x_1, \bar{x}_1}(t) \right\| \\ &\leqslant Ch^2 \, |||\xi|||_{C^4(I, L^2(\tilde{I}))}, \\ \|M_5(t)\| &\leqslant C\tau \, \|\Delta \xi_t(t)\| \leqslant C\tau \left|\left|\left| 2 \frac{\partial \xi}{\partial t} \right|\right|\right| \\ &\leqslant C\tau \left( \left|\left|\left| \frac{\partial \xi}{\partial t} \right|\right|\right|_{C^2(I, L^2(\tilde{I}))} + \left|\left|\left| \frac{\partial \xi}{\partial t} \right|\right|\right|_{C(I, H^2(\tilde{I}))} \right), \\ \|M_6(t)\| &\leqslant Ch^2 \left\| \frac{\partial^2 \psi}{\partial x_1^2}(t) - \psi_{x_1, \bar{x}_1}(t) \right\| \\ &\leqslant Ch^2 \, |||\psi|||_{C^2(I, L^2(\tilde{I}))}.\end{aligned}$$

Finally, an argument as in the proof of Theorem 1 leads to the following conclusion.

THEOREM 2. *Let conditions* (1) *and* (2) *of Theorem* 1 *hold. Also let* $\mu > 0$ *and* $r > 0$, *and assume that*

$$\begin{aligned}&\xi \in C(0, T; H^{2+r}(\Omega) \cap C^4(I, L^2(\tilde{I})) \cap H^{1/2+r}(I, H^{\mu+1}(\tilde{I})) \\ &\qquad \cap H^{3/2+r}(I, H^{\mu}(\tilde{I})) \cap H^{7/2+r}(I, L^2(\tilde{I}))), \\ &\frac{\partial \xi}{\partial t} \in C(0, T; C(I, H^2(\tilde{I})) \cap C^2(I, L^2(\tilde{I})) \\ &\qquad \cap H^{1/2+r}(I, H^1(\tilde{I})) \cap H^{3/2+r}(I, L^2(\tilde{I})), \\ &\frac{\partial^2 \xi}{\partial t^2} \in C(0, T; C(I, L^2(\tilde{I}))), \\ &\psi \in C(0, T; H^{2+r}(\Omega) \cap C^2(I, L^2(\tilde{I})) \\ &\qquad \cap H^{1/2+r}(I, H^{\mu+1}(\tilde{I})) \cap H^{3/2+r}(I, H^{\mu}(\tilde{I}))).\end{aligned}$$

*Then for all* $t \leqslant T$ *we have*

$$\|\xi(t) - \eta^{(N)}(t)\|^2 \leqslant Cb^*(\tau^2 + h^4 + N^{-2\mu}),$$

*where* $b^*$ *is a positive constant dependent upon the norms appearing in the estimates of the terms* $\|M_q(t)\|$.

## VII. Discussion

The spectral-difference method is better than the full difference method. But the accuracy is still limited by the order of the difference approximation as shown in Tables I and II. If we were to use Chebyshev methods in the direction of non-periodicity, we could solve the same problem with a tremendous gain in accuracy.

If we use the pseudospectral-difference method to solve (1.1), then we can save computation, especially for the nonlinear convective term. But for the second equation of (1.1) it is easy to use the spectral-difference method. We shall report on a comparison of these methods in a future paper.

By a skew-symmetric decomposition of the nonlinear convection terms, we can obtain better numerical results than by the more conventional form. But a little more computation is needed for calculating Fourier coefficients of the convection term. If we use the pseudospectral-difference method, then the computations are nearly the same either way.

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